数理解析研究所講究録
第 1906 巻 2014 年 162-171

Operator Algebraic Shannon's Interpretation for Entropy-preserving Stochastic Averages

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Abstract

We study various relations of ρ and Φ from the view point of the von Neumann entropy. Here ρ and Φ are a state and a unital positive Tr-preserving linear map on the algebra $M_n(\mathbb{C})$ of $n\times n$ complex matrices respectively. For the state ρ and the new state $\rho\circ\Phi$ arising as the composition, we show among the others that these two states have the same value of the von Neumann entropy if and only if Φ behaves for ρ as some automorphism of $M_n(\mathbb{C})$.

1 Introduction

Shannon([8, p.395 4.]) denotes as the followings: If we perform any "averaging " operation on the $\{p_i\}_{i=1,\cdots,n}$ of the form

$$p_i' = \sum_j a_{ij}p_j$$

(where $p_i\geq 0$, $\sum_i p_i = 1$ and $a_{ij}\geq 0$, $\sum_i a_{ij} = \sum_j a_{ij} = 1$), the entropy H increases (except in the special case where this transformation amounts to no more than a permutation of the p_i with H of course remaining the same).

This means the followings: The entropy $H(\lambda)$ of a probability vector $\lambda = (\lambda_1,\cdots,\lambda_n)$ and the entropy $H(\lambda b)$ of the probability vector λb for a bistachastic matrix $b=[b_{ij}]$ are always in the relation that $H(\lambda)\leq H(\lambda b)$ and the two values are equal if and only if the bistochastic matrix b behaves just as a permutation σ, i.e. $\lambda b = (\lambda_{\sigma(1)},\cdots,\lambda_{\sigma(n)})$.

Replacing a probability vector $\lambda\in\mathbb{R}^n$ (resp. a bistachastic matrix b) to a state ρ of $M_n(\mathbb{C})$ (resp. a unital positive Tr-preserving linear map Φ on

$M_n(\mathbb{C})$), we show, among the others, that the von Neumann entropy $S(\cdot)$ increases by performing any Φ on ρ (except in the special case where this transformation amounts to no more than an automorphism α of state ρ with $S(\cdot)$ of course remaining the same).

2 Notations, terminologies and basic facts

The main tool is the *entropy function* η defined on the interval $[0, 1]$ by

$$\eta(t) = -t \log t \quad (0 < t \leq 1) \quad \text{and} \quad \eta(0) = 0.$$

The η is *strictly concave*, i.e. for two k-tuples of real numbers $\{s_i\}$, $\{t_i\}$ such that $s_i \geq 0, t_i > 0, \sum_{i=1}^k t_i = 1$, it holds that

$$\sum_{i=1}^{k} t_i \eta(s_i) \leq \eta(\sum_{i=1}^{k} t_i s_i),$$

and the equality holds if and only if $s_i = s_j$ for all i, j.

Moreover, η is *strictly operator-concave*, i.e. the similar relations hold by replacing $\{s_i\}_i$ to any bounded self-adjoint operators $\{x_i\}_i$ with spectra in [0,1], i.e.

$$\sum_{i=1}^{k} t_i \eta(x_i) \leq \eta(\sum_{i=1}^{k} t_i x_i)$$

and the equality implies that $x_i = x_j$ for all i, j. (see for example [4, B], [5, 6]).

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a probability vector in $\mathbb{R}^n$, i.e. $\lambda_i \geq 0$ for all i and $\sum_i \lambda_i = 1$. The *Shannon entropy* $H(\lambda)$ for λ is given as

$$H(\lambda) = \eta(\lambda_1) + \cdots + \eta(\lambda_n).$$

It holds always that $H(\lambda) \leq \log n$ and $H(\lambda) = \log n$ if and only if $\lambda_i = 1/n$ for all $i = 1, \cdots, n$.

Throughout this note, let H be an n-dimensional Hilbert space. We denote by M the algebra $B(H)$ of linear operators on H so that M is isomorphic to $M_n(\mathbb{C})$, i.e. the C^*-algebra of $n \times n$ matrices over the complex field $\mathbb{C}$. By Tr we mean the standard trace of M such that $\text{Tr}(e) = 1$ for every minimal projection e in M.

Every positive linear functional ϕ on M is of the form $\phi(x) = \mathrm{Tr}(D_\phi x)$, $(x \in M)$ for a unique positive element $D_\phi \in M$ which is called the *density operator* or *density matrix* of ϕ. If ρ is a state of M, then the density matrix D_ρ is a positive operator in M such that $\mathrm{Tr}(D_\rho) = 1$.

By using the eigenvalue list $\{\lambda_1, \cdots, \lambda_n\}$ of D_ρ, the *von Neumann entropy* $S(\rho)$ and $S(D_\rho)$ for ρ and D_ρ are defined by

$$S(\rho) = S(D_\rho) = \sum_{i=1}^{n} \eta(\lambda_i).$$

3 The von Neumann entropy and stochastic averages

Our purpose of this note is to give a generalized version of Shannon's interpretation for entropy-preserving stochastic averages of probability vectors to the framework of von Neumann entropy for states on $M_n(\mathbb{C})$.

In this section, we discuss the Shannon's interpretation in the framwork of the von Neumann entropy as follows:

Replace a probability vectors λ to a state ρ of $M_n(\mathbb{C})$, a bistachastic matrix b to a unital positive trace preserving map Φ on $M_n(\mathbb{C})$, and the Shannon entropy $H(\cdot)$ to the von Neumann $S(\cdot)$, then a permutation changes into an automorphism α of $M_n(\mathbb{C})$, i.e., $S(\rho \circ \Phi) = S(\rho)$ if and only if $\rho \circ \Phi = \rho \circ \alpha$ for some automorphism α.

3.1 The pair $\{\rho, \Phi\}$ of state ρ and positive map Φ.

Let ρ be a state of $M_n(\mathbb{C})$. We denote by D_ρ the density matrix of ρ, i.e., D_ρ is a positive operator in $M_n(\mathbb{C})$ which satisfies that

$$\mathrm{Tr}(D_\rho) = 1 \quad \text{and} \quad \rho(x) = \mathrm{Tr}(D_\rho x) \text{ for all } x \in M_n(\mathbb{C}).$$

Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a positive unital Tr preserving map. Then $\Phi(D_\rho)$ is a operator in $M_n(\mathbb{C})$ and $\mathrm{Tr}(\Phi(D_\rho)) = 1$.

In order to see the state whose density matrix is $\Phi(D_\rho)$, we need the system of the Hilbert-Schmidt inner product of $M_n(\mathbb{C})$: The inner product and the norm are given by

$$< x, y > = \operatorname{Tr}(y^* x) \quad \text{and} \quad \|x\|_2 = (\operatorname{Tr}(x^* x))^{1/2} \quad \text{for} \quad x, y \in M_n(\mathbb{C}).$$

The *-preserving map Φ induces the adjoint map $\Phi^* : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with respect to this $< \cdot, \cdot >$ by

$$\operatorname{Tr}(y\Phi^*(x)) = \operatorname{Tr}(\Phi(y)x) \quad x, y \in M_n(\mathbb{C}). \tag{3.1}$$

Since Φ is positive, it follows that Φ^* is positive, and $\rho \circ \Phi^*$ is a state by the property that Tr Φ = Tr.

The $\Phi(D_\rho)$ is the density matrix of this state $\rho \circ \Phi^*$ because

$$\rho \circ \Phi^*(x) = \operatorname{Tr}(D_\rho \Phi^*(x)) = \operatorname{Tr}(\Phi(D_\rho)x), \quad (x \in M_n(\mathbb{C})).$$

We let the set of eigenvalues of D_ρ and $\Phi(D_\rho)$ be

$$\lambda = (\lambda_1, \cdots, \lambda_n) \quad \text{and} \quad \mu = (\mu_1, \cdots, \mu_n), \tag{3.2}$$

respectively. Here we arrange them always in a decereasing order, i.e.,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n. \tag{3.3}$$

We let $\{e_1, \cdots, e_n\}$ (resp. $\{p_1, \cdots, p_n\}$) be mutually orthogonal minimal projections, which gives the spectral decomposition of D_ρ (resp. $\Phi(D_\rho)$):

$$D_\rho = \sum_{i=1}^{n} \lambda_i e_i \quad (\text{resp.} \quad \Phi(D_\rho) = \sum_{j=1}^{n} \mu_j p_j). \tag{3.4}$$

We denote by A (resp. B) the maximal abelian subalgebra of $M_n(\mathbb{C})$ which is generated by the projections $\{e_1, \cdots, e_n\}$ (resp. $\{p_1, \cdots, p_n\}$).

3.1.1 The unitary $u_{(\rho,\Phi)}$ arising from the pair $\{\rho, \Phi\}$.

In these setting, a unitary $u_{(\rho,\Phi)}$ appears and satisfies the following relation:

$$u_{(\rho,\Phi)}\ e_i \ = p_i\ u_{(\rho,\Phi)}, \quad \text{for all} \quad i = 1, \cdots, n \tag{3.5}$$

3.1.2 Bistochastic matrix $b_\rho(\Phi)$ for the pair $\{\rho, \Phi\}$

Definition 3.1. We define a matrix $b_\rho(\Phi)$ by the formula

$$b_\rho(\Phi)_{ij} = \mathrm{Tr}(\Phi(e_i)p_j), \quad (1 \le i \le n, 1 \le j \le n). \tag{3.6}$$

Lemma 3.2. *Let ρ be a state of $M_n(\mathbb{C})$, and let Φ be a unital positive* Tr-*preserving map on $M_n(\mathbb{C})$. Let λ and μ be the probability vectors of the eigenvalues of D_ρ and $\Phi(D_\rho)$ respectively. Then the followings hold:*

(1) *The $b_\rho(\Phi)$ is a bistochastic matrix.*

(2) *The probability vector $\lambda \in \mathbb{R}^n$ is transposed to the probability vector $\mu \in \mathbb{R}^n$ by the matrix $b_\rho(\Phi)$:*

$$\lambda b_\rho(\Phi) = \mu.$$

Definition 3.3. For each j, we set

$$I_j = \{i : b_\rho(\Phi)_{ij} \neq 0\}.$$

Lemma 3.4. *Let ρ be a state of $M_n(\mathbb{C})$, and let Φ be a unital positive* Tr-*preserving map on $M_n(\mathbb{C})$. Assume that $S(\Phi(D_\rho)) = S(D_\rho)$. Then, for each j, we have that*

$$\lambda_i = \lambda_k \quad \textit{for all} \quad i, k \in I_j.$$

Under the assumption that $S(\Phi(D_\rho)) = S(D_\rho)$, we denote the constant λ_i for $i \in I_j$ in the above Lemma by $\lambda^{(j)}$. Remark that each I_j is a non empty set because $b_\rho(\Phi)$ is a bistochastic matrix, and

$$\lambda^{(j)} = \frac{\sum_{i \in I_j} \lambda_i}{|I_j|} = \lambda_k \quad \text{for all} \quad k \in I_j.$$

Theorem 3.5. *Let ρ be a state of $M_n(\mathbb{C})$ and let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital positive* Tr-*preserving map. Then the followings are equivalent:*

(i) $S(\rho \circ \Phi^*) = S(\rho)$, *i.e.* $S(\Phi(D_\rho)) = S(D_\rho)$.

(ii) $\lambda = \mu b_\rho(\Phi)^T$, *i.e.* $\lambda = \lambda b_\rho(\Phi) b_\rho(\Phi)^T$
where $\{\}^T$ denotes the transpose.

(iii) $\lambda_i = \mu_i$ *for all* $i = 1, \cdots, n$.

(iv) *The unitary $u_{(\rho,\Phi)}$ satisfies that $\Phi(D_\rho) = u_{(\rho,\Phi)}\, D_\rho\, u^*_{(\rho,\Phi)}$.*

Remark 3.6. If a state ρ is the normalized trace Tr/n, then the density matrix is I_n/n so that the all statements in the above theorem are trivial.

Remark 3.7. In the case of $n = 2$, if $\lambda bb^T = \lambda$ for a bistochastic matrix b, then b is the nontrivial permutation.

In fact, let $\lambda = (\lambda_1, \lambda_2)$. Every 2×2 bistochastic matrix $b = (b_{ij})$ is written as $b_{11} = b_{22} = b_1$ for some $0 \leq b_1 \leq 1$ and $b_{12} = b_{21} = b_2 = 1 - b_1$. If $\lambda bb^T = \lambda$, then $\lambda_1 = \lambda_1(b_1^2 + b_2^2) + 2\lambda_2 b_1 b_2$ and $\lambda_2 = \lambda_2(b_1^2 + b_2^2) + 2\lambda_1 b_1 b_2$. This implies that $\lambda_1 b_1(2b_1 - 2) + b_1(1 - b_1) = 0$. Hence if $\lambda_1 = 0$ then $b_1 = 0$ or $b_1 = 1$, which means that b is permutation matrix. Assume that $\lambda_1 \neq 0$. We may omit the case $b_1 = 1$ and so we assume $b_1 \neq 1$, Then $\lambda_1 = 1/2$ or $b_1 = 0$. As we omit that λ is the trivial case so that $b_1 = 0$, i.e. b is the non-trivial permutation.

Corollary 3.8. *Assume that $S(\Phi(D_\rho)) = S(\rho)$ holds for the pair $\{\rho, \Phi\}$ of a state ρ of $M_n(\mathbb{C})$ and a unital positive* Tr*-preserving map Φ on $M_n(\mathbb{C})$. Then*

$$< \Phi(D_\rho), \Phi(e_k) >=< D_\rho, e_k > \quad \textit{for all } k.$$

A linear map Φ on $M_n(\mathbb{C})$is said to be *2-positive* if $\Phi \otimes id$ (the tensor product of Φ and the identity map on $M_2(\mathbb{C})$) on $M \otimes M_2(\mathbb{C})$ is positive. It is well known that if Φ is 2-positive, then Φ^* is 2-positive and the so-called Kadison-Schwartz inequality holds [2], (cf. [4, 5, 6]):

$$\Phi^*(x^*)\Phi^*(x) \leq \Phi^*(x^*x), \quad (x \in M).$$

Corollary 3.9. *Let ρ be a state of $M_n(\mathbb{C})$, and let Φ be a unital positive* Tr*-preserving map on $M_n(\mathbb{C})$.*

If Φ is 2-positive, then the following conditions are equivalent:
(i') $S(\Phi(D_\rho)) = S(D_\rho)$
(iv) $\Phi(D_\rho) = uD_\rho u*$ *for some unitary u.*
(v) $\Phi^*\Phi(D_\rho) = D_\rho$

Related results are obtained in [7] and [3].

Example 3.10. The conditional expectation conditioned by Tr/n is a most typical example of unital completely posive (so that 2-positive) Tr-preserving linear map of $M_n(\mathbb{C})$. Let E be such a conditional expectation of $A = M$ to a C^*-subalgebra B with $1_A = 1_B$. Then

$$S(E(D_\rho)) = S(D_\rho) \quad \text{if and only if} \quad D_\rho \in B.$$

In fact, the conditional expectation E satisfies that $E^*E = E$. By combining this fact with Corollary 3.7, we have that $S(E(D_\rho)) = S(\rho)$ if and only if $D_\rho = E^*E(D_\rho) = E(D_\rho)$ which means that $D_\rho \in B$.

3.2 Relations among various entropies

The weighted entropy $H^\lambda(b)$ and $H_\lambda(b)$ for a bistochastic matrix $b = [b_{ij}]$ with respect to a probability vector $\lambda = (\lambda_1, \cdots, \lambda_n)$ are defined in [10] by the following forms:

$$H^\lambda(b) = \sum_{j=1}^n \lambda_j \sum_{k=1}^n \eta(b_{jk}) \quad \text{and} \quad H_\lambda(b) = \sum_{k=1}^n \lambda_k \sum_{j=1}^n \eta(b_{jk}).$$

In the case where $\lambda_i = 1/n$ for all i, these are denoted by $H(b)$ simply :

$$H(b) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \eta(b_{ij}).$$

Definition 3.11. We let

$$J_\lambda = \{k; \lambda_k \neq 0\}.$$

Since Φ is positive and Tr-preserving, each $\Phi(e_i)$ is a density matrix which induces the state ρ_i given by $\rho_i(x) = \mathrm{Tr}(\Phi(e_i)x)$ for all $x \in M_n(\mathbb{C})$.

Definition 3.12. Now we pick up the following constant $S_\rho(\Phi)$ which is a convex combination of the entropies $\{S(\Phi(e_i)); i = 1, \cdots, n\}$ with respect to the eigenvalues of the density matrix D_ρ:

$$S_\rho(\Phi) = \sum_{i=1}^n \lambda_i S(\Phi(e_i)) = \sum_{i=1}^n \lambda_i S(\rho_i).$$

The algebra B is a typical von Neumann subalgebra of the I_n-factor $M_n(\mathbb{C})$ and there exists always a positive linear map E_B from $M_n(\mathbb{C})$ onto N such that $aE(x)b = E(axb)$ for all $x \in M$ and $a, b \in B$ which is called conditional expectation of $M_n(\mathbb{C})$ onto B.

Lemma 3.13. *Let E_B be the conditional expectation of $M_n(\mathbb{C})$ onto B. Then*

$$E_B(\Phi(e_i)) = \sum_{j=1}^{n} b_\rho(\Phi)_{ij} p_j \quad \textit{for each} \quad i.$$

so that

$$H^\lambda(b_\rho(\Phi)) = \sum_{i=1}^{n} \lambda_i S(E_B(\Phi(e_i))).$$

Theorem 3.14. *Let ρ be a state of $M_n(\mathbb{C})$, and let Φ be a unital positive* Tr*-preserving map on $M_n(\mathbb{C})$.*

Then the following relations hold for the weighted entropies of the bistochastic matrix $b_\rho(\Phi)$ with respect to the eigenvalue list $\lambda = (\lambda_1, \cdots, \lambda_n)$ of D_ρ, $S_\rho(\Phi)$ and the eigenvalue list $\mu = (\mu_1, \cdots, \mu_n)$ of $\Phi(D_\rho)$:

(1)

$$S_\rho(\Phi) \ \leq H^\lambda(b_\rho(\Phi)) \leq S(\rho \circ \Phi^*) \ \leq S(\rho) + S_\rho(\Phi).$$

(2) $S_\rho(\Phi) = H^\lambda(b_\rho(\Phi))$ *if and only if* $\Phi(e_i) \in B$ *for all* $i \in J_\lambda$.

(3) $H^\lambda(b_\rho(\Phi)) = S(\rho \circ \Phi^*)$ *if and only if*

$$(\mu_1, \cdots, \mu_n) = (b_\rho(\Phi)_{i1}, \cdots, b_\rho(\Phi)_{in}) \quad \textit{for all} \quad i \in J_\lambda.$$

(4) $S_\rho(\Phi) = S(\rho \circ \Phi^*)$ *if and only if* $\Phi(D_\rho) = \Phi(e_i)$ *for every* $i \in J_\lambda$.

(5) $S(\rho \circ \Phi^*) = S(\rho) + S_\rho(\Phi)$ *if and only if the* ρ *is a pure state.*

Remark 3.15. The above statement (3) says that $H^\lambda(b_\rho(\Phi)) = S(\Phi(D_\rho))$ if and only if $b_\rho(\Phi)$ has the following form:

$$b_\rho(\Phi) = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ b_\rho(\Phi)_{k1} & b_\rho(\Phi)_{k2} & \cdots & b_\rho(\Phi)_{kn} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ b_\rho(\Phi)_{n1} & b_\rho(\Phi)_{n2} & \cdots & b_\rho(\Phi)_{nn} \end{bmatrix}.$$

Here $k = |J_\lambda| + 1$ for the cardinality $|J_\lambda|$ of J_λ.

Corollary 3.16. *If $\lambda_i \neq 0$ for all $i = 1, \cdots, n$ and if Φ satisfies that $H^\lambda(b_\rho(\Phi)) = S(\rho \circ \Phi^*)$, then*

$$\mu_j = b_\rho(\Phi)_{ij} = \frac{1}{n} \quad \textit{for all} \quad i, j = 1, \cdots, n,$$

so that $\rho \circ \Phi^*$ is the normalized trace Tr/n and $S(\rho \circ \Phi^*) = \log n$.

Remark 3.17. **(A connection with Hadamard matrix).** A bistochastic matrix b is said to be *unistochastic* if it is induced from some unitary matrix u by that $b_{i,j} = |u_{i,j}|^2$ for all $i, j = 1, \cdots, n$. A $n \times n$ unitary matrix u is called a *Hadamard matrix* if $|u_{i,j}| = 1/\sqrt{n}$ for all $i, j = 1, \cdots, n$.

The above corollary means that if D_ρ has only non-zero eigenvalues, (i.e., $\lambda_i \neq 0$ for all i) and if $H^\lambda(b_\rho(\Phi)) = S(\Phi(D_\rho))$ then $b_\rho(\Phi)$ is a unistochastic matrix induced from a Hadamard matrix.

Example 3.18. Here, we give some examples.

(1) If ρ is a pure state, then the four kinds constants satisfy that

$$S(\rho) = 0 \quad \text{and} \quad S_\rho(\Phi) = H^\lambda(b_\rho(\Phi)) = S(\rho \circ \Phi^*)$$

for all positive unital Tr-preserving map Φ.

(2) If Φ is a *-isomorphism, then for all state ρ the followings hold:

$$S_\rho(\Phi) = 0 \quad \text{and} \quad S(\rho \circ \Phi^*) = S(\Phi(D_\rho)) = S(D_\rho) = S(\rho).$$

In fact, if Φ ia a *-isomorphism, then $\Phi(e)$ is a minimal projection for a minimal projection e, so that $S_\rho(\Phi) = \sum_{i=1}^n \lambda_i S(\Phi(e_i)) = 0$ and of course $S(\Phi(D_\rho)) = S(D_\rho)$.

(3) If Φ is a unital positive Tr-preserving map to the center $\mathbb{C}1_M$ of $M_n(\mathbb{C})$, then

$$S_\rho(\Phi) = H^\lambda(b_\rho(\Phi)) = S(\rho \circ \Phi^*) = \log n \quad \text{for all state } \rho.$$

In fact, for each i, put $\Phi(e_i) = \alpha_i 1_M$ for $\alpha_i \in \mathbb{C}$, then $1 = \mathrm{Tr}(\mathrm{e_i}) = \mathrm{Tr}(\Phi(\mathrm{e_i})) = \alpha_\mathrm{i}\mathrm{Tr}(\Phi(1_\mathrm{M})) = \alpha_\mathrm{i}\mathrm{n}$ so that $\Phi(e_i) = 1_M/n$. This implies that $S_\rho(\Phi) = \sum_i \lambda_i S(1_M/n) = \mathrm{Tr}(\eta(1_M/n)) = \log n$. Remember that in general $S_\rho(\Phi) \leq H^\lambda(b_\rho(\Phi)) \leq S(\rho \circ \Phi^*) \leq \log n$ for all state ρ. Hence we have the equality.

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